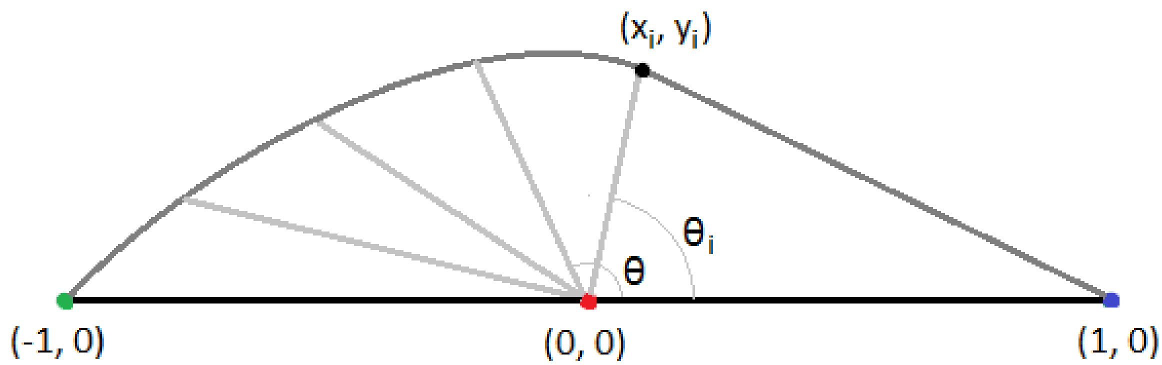
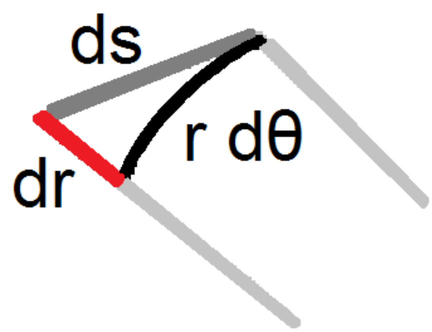
A submarine is sitting exactly halfway between a ship and its home port. The submarine submerges, and the ship will have no further information on its position. To sink the ship, the sub will have to be directly under it. If the ship is fast enough, it will be able to set a wide course around the sub and reach the port. How much faster does the ship have to be to guarantee it can avoid the sub and get home?

Answer- The ship needs to be about **2.3325** times faster than the sub.

Let the ship's speed be k times the speed of the submarine. If k is 2 or less, the sub can head directly toward the port and beat the ship there. On the other hand, if k is greater than π, the ship can make a semicircle around the sub to reach the port while remaining out of the sub's range. So we know $2 < k_{min} < \pi$. There is a limiting case where the ship plots a course around the sub such that whatever angle θ the sub tries to intercept (after some angle θ_i), they arrive at the same time. [We can place the ship at (1, 0) without loss of generality.]



This gives us a polar curve terminating at (-1, 0) and a straight line from (1, 0) to pick it up. Note the two pieces have the same slope at (x_i, y_i). The polar curve has the following property: as θ increases, the arc length increases k times as much as r increases- ds = k dr.



This relation leads to the differential equation:

$$ds = \sqrt{dr^2 + (r\,d\theta)^2} = k\,dr \qquad (1)$$

$$dr^2 + r^2\,d\theta^2 = k^2\,dr^2$$

$$dr\sqrt{k^2 - 1} = r\,d\theta$$

$$\frac{dr}{d\theta} = \frac{r}{\sqrt{k^2 - 1}}$$

(2)

Since $\sqrt{k^2 - 1}$ is a constant, this gives $r(\theta) = Ce^{\frac{\theta}{\sqrt{k^2-1}}}$. We know r(π) = 1, so C = $e^{\frac{-\pi}{\sqrt{k^2-1}}}$, and we obtain:

$$\mathrm{r}(\theta) = e^{\frac{\theta - \pi}{\sqrt{k^2-1}}}$$

(3)

This gives us a family of curves parameterized by k. We want the curve where the sub and the ship would reach (x_i, y_i) at the same time. This means that:

$$e^{\frac{\theta_i - \pi}{\sqrt{k^2-1}}} = \frac{\sqrt{(1 - x_i)^2 + y_i{}^2}}{k}$$

(4)

The tangency requirement leads to the equation:

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_i} = \frac{-y_i}{1 - x_i}$$

But

$$\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + \mathrm{r}'\cos\theta} = \frac{r\cos\theta + \frac{r}{\sqrt{k^2-1}}\sin\theta}{-r\sin\theta + \frac{r}{\sqrt{k^2-1}}\cos\theta} = \frac{\cos\theta + \frac{\sin\theta}{\sqrt{k^2-1}}}{-\sin\theta + \frac{\cos\theta}{\sqrt{k^2-1}}}$$

so

$$\frac{-y_i}{1 - x_i} = \frac{\cos\theta_i + \frac{\sin\theta_i}{\sqrt{k^2-1}}}{-\sin\theta_i + \frac{\cos\theta_i}{\sqrt{k^2-1}}}$$

(5)

A good guess of $\theta_i = \frac{\pi}{2}$ collapses both equations (4) and (5) to

$$e^{-\frac{\pi}{2\sqrt{k^2-1}}} = \frac{1}{\sqrt{k^2-1}}$$

[The algebra is painful, but taking (4), (5), and $y_i > 0$ gives $\cos\theta_i = 0$.] Solving this equation numerically gives k = 2.3325…, so if the ratio of the ship’s speed to the sub’s is greater than that, the ship will be able to plot this course and avoid the sub.